ON THE BOXICITY AND CUBICITY OF A GRAPH

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1. Introduction

Suppose A is a finite set of *points*, I is a reflexive, symmetric binary relation of *adjacency* on A, and G denotes the *graph* (A, I). All graphs in the following will be finite and have loops at each point as here. Several authors [1–3] have studied the notion of an *interval graph*, i.e., a graph G for which there is an assignment to each point x in A of an interval $N(x)$ on the real line such that for all $x, y, \in A$,

$$xIy \leftrightarrow N(x) \cap N(y) \neq \varnothing. \tag{1}$$

In the following, we consider the problem of finding an assignment satisfying (1) where the $N(x)$ are taken to be "boxes" in Euclidean n-space E^n, i.e., generalized rectangles with sides parallel to the coordinate axes.

Simultaneously, we consider a closely related problem. In [4], we studied the notion of *indifference graph*. This is a graph (A, I) for which there is a real-valued function f on A so that for all $x, y \in A$,

$$xIy \leftrightarrow d(f(x), f(y)) \leq 1, \tag{2}$$

where d is the usual metric on the real line, i.e., absolute value. A natural generalization is to ask for a representation (2) where f takes values in E^n and d is an appropriate metric on E^n. To maintain an analogy with the generalization of the interval graph problem, it is convenient to take the "product metric,"

$$d(\langle x_1, x_2, \ldots, x_n\rangle, \langle y_1, y_2, \ldots, y_n\rangle) = \max_i |x_i - y_i|,$$

because then the representation (2) corresponds exactly to a representation (1) with the $N(x)$ all closed[1] cubes of side-length 1. (Choice of this metric can also be motivated on economic and psychological grounds.)

These two representation problems will lead to two notions of dimension of a graph, the boxicity and cubicity,[2] and it will be our aim to establish a sharp upper bound for dimension in each case as a function of the number of points. We start in the next section with the second representation problem. The development for the first will be entirely analogous, and perhaps a little simpler.

2. The Cubicity

With this introduction, we now propose to study what graphs (A, I) can be *embedded into n-space* in the sense that there is a function $f: A \to E^n$ satisfying (2), where d is the product metric. It is somewhat easier to formulate the problem in terms of the coordinate functions $f_1, f_2, \ldots, f_n$ of f. Thus we ask: when do there exist real-valued functions $f_1, f_2, \ldots, f_n$ on A so that for all $x, y \in A$,

$$xIy \leftrightarrow (\forall i \leq n)[|f_i(x) - f_i(y)| \leq 1]. \tag{3}$$

It is convenient here to make one slight convention which is quite natural and makes the results a little neater to state. It seems fair to speak of a graph (A, I) as embeddable into 0-space in the above sense if and only if all points in A are adjacent, i.e., (A, I) is complete.

We begin by noting the not too surprising result that every graph with n points is embeddable into n-space in the sense of Eq. (3). To see this, simply list A as $a_1, a_2, \ldots, a_n$, and define for $i = 1, 2, \ldots, n$,

$$f_i(x) = \begin{cases} 0 & \text{if} \quad x = a_i \\ 1 & \text{if} \quad xIa_i,\ x \neq a_i \\ 2 & \text{if} \quad {\sim}xIa_i. \end{cases}$$

This observation permits us to define, motivated by the intersection interpretation, the *cubicity*[3] of a graph G, cub G, as the smallest n so that G is embeddable into n-space. Each graph has finite cubicity, and indeed a graph of n points has cubicity at most n.

We close this section with a few simple but basic remarks about embeddability and cubicity. In particular, it will be helpful to study the intersection of two graphs with the same point set. We note readily that if $n \geq 1$, a graph G

[1] Boxes are not necessarily closed, though it is not hard to show that if a representation (1) is attainable with boxes in E^n, it is attainable with closed boxes in E^n.

[2] See acknowledgments.

[3] More precisely, the unit cubicity.

is embeddable into n-space if and only if G is the intersection of n indifference graphs. For, each coordinate function gives a representation (2). It follows easily that if $G = H \cap K$, then cub $G \leq$ cub $H +$ cub K.

Following [4], it is convenient to introduce an equivalence relation E on the points of the graph $G = (A, I)$. This is defined by $xEy \leftrightarrow (\forall z)(xIz \leftrightarrow yIz)$. Note that since our graphs are reflexive, equivalence implies adjacency. Thus, two points are equivalent if and only if they have the same "closed neighborhoods." The relation E is significant because if aEb, then cub $G =$ cub $G - a$ $=$ cub $G - b$, where $G - x$ is the subgraph[4] generated by points different from x. (Map a and b onto the same point in n-space.) More generally, cub $G =$ cub G/E, where G/E is the graph obtained from G by cancelling out E.

3. The Cubicity of the Complete Partite Graphs

Having proved that every graph of n points can be embedded into n-space, we are interested in solving the following extremum problem: given n, what is the smallest k so that we can embed all graphs with n points into k-space? Put another way, what is the maximum cubicity $c(n)$ of all graphs with n points?

To study the function $c(n)$, it turns out to be extremely useful to know the cubicity of the so-called complete partite graphs, for the maximum cubicity is actually attained for each n in such a graph. We shall devote this section to calculating an explicit cubicity formula for the complete partites. A graph (A, I) will be called *complete partite* if A can be written as the disjoint union of nonempty classes so that no lines (except loops) within a class occur and all lines between points of different classes occur. We shall denote by $K(n_1, n_2, \ldots, n_p)$ the graph consisting of p such classes, containing $n_1, n_2, \ldots,$ n_p points, respectively ($n_i > 0$). Of particular interest are the graphs $K(1, n)$. We shall for simplicity denote $K(1, n)$ by $S(n)$, and call it a *star of n vertices*. The singleton point will be called the *center* of the star and the remaining n points the *vertices*. To calculate the cubicity of the graph $G = K(n_1, n_2, \ldots, n_p)$, we shall first calculate the cubicity of the star $S(n)$, for each n, and then express cub G in terms of the cubicities of the stars $S(n_i)$.

THEOREM 1. *$S(n)$ is embeddable into k-space if and only if $n \leq 2^k$. Thus,* cub $S(n) = [\log_2(2n - 1)]$, *where $[x]$ is the greatest integer in x.*

[4] Subgraph will always mean "generated" subgraph, i.e., all adjacent lines (edges) are included.

PROOF: Suppose $n \leq 2^k$. If $k = 0$, then $n = 1$ and the graph is complete. If $k > 0$, embed $S(n)$ into k-space by sending the center into the origin $\langle 0, 0, \ldots, 0\rangle$ and the vertices into points of the form $\langle \pm 1, \pm 1, \ldots, \pm 1\rangle$.

Next, suppose $n > 2^k$. We show by induction on k that $S(n)$ cannot be embedded into k-space. If $k = 0$, then $n > 2^k$ implies $S(n)$ is not complete, and so not embeddable into k-space. If $k = 1$, then $n > 2^k$ implies that $S(n)$ contains $S(3) = K(1, 3)$ as a subgraph. But $S(3)$ is not embeddable into 1-space, as is easily verified directly or from the results of [4]. Thus, $S(n)$ is not embeddable into 1-space. We now assume the result for $k \geq 1$ and prove it for $k + 1$. Suppose $n > 2^{k+1}$ and suppose by way of contradiction that $f_1, f_2, \ldots, f_{k+1}$ is an embedding of $S(n)$ into $(k + 1)$-space. Let b denote the center of $S(n)$ and $a_1, a_2, \ldots, a_n$ its vertices, and let

$$A_1 = \{a_i : f_{k+1}(a_i) \geq f_{k+1}(b)\}$$

and

$$A_2 = \{a_i : f_{k+1}(a_i) \leq f_{k+1}(b)\}.$$

We may suppose that A_1 has $t > 2^k$ elements. But then if g_i is the restriction of f_i to $A_1 \cup \{b\}$, it follows that $g_1, g_2, \ldots, g_k$ is an embedding into k-space of the star $S(t)$ whose center is b and whose vertices are the elements of A_1. This contradicts the inductive assumption.

THEOREM 2. *Suppose* $G = K(n_1, n_2, \ldots, n_p)$. *Then,*

(a) *if* $p > 1$,

$$\text{cub } G = \sum_{j=1}^{p} \text{cub } S(n_j),$$

(b) *if* $p = 1$,

$$\text{cub } G = \begin{cases} 1 & \text{if } n_p > 1 \\ 0 & \text{if } n_p = 1. \end{cases}$$

PROOF: Part (b) is trivial. To prove (a), let $G = (A, I)$ and let $C_1, C_2, \ldots, C_p$ be the classes of $n_1, n_2, \ldots, n_p$ points, respectively. Define H_j to be the graph which is obtained from G by adding all lines (edges) within classes different from C_j. Note that $G = \bigcap_{j=1}^{p} H_j$. Also, cub $H_j =$ cub $S(n_j)$, since all points of $A - C_j$ are equivalent[5] in H_j. This gives one-half of the desired formula, namely an inequality in one direction:

$$\text{cub } G \leq \sum \text{cub } H_j = \sum \text{cub } S(n_j). \tag{4}$$

To get the inequality in the other direction, let $n =$ cub G and note first that G has cubicity 0 (i.e., G is complete) if and only if each n_j is 1, which is the case if and only if $\sum$ cub $S(n_j) = 0$. This proves the second inequality in

[5] Equivalence will always refer to the relation E defined previously.

the case $n = 0$. Next, suppose $n \geq 1$. We may write $G = G_1 \cap G_2 \cap \ldots \cap G_n$, where each $G_i = (A, I_i)$ is an indifference graph. If $n_j > 1$, let U_j be the collection of all I_i so that I_i is missing a line between two points of C_j, and define a graph $K_j = \bigcap \{G_i : I_i \in U_j\}$. Then, K_j is the same as the corresponding graph H_j described above. This follows easily from the observation that each I_i is in one and only one U_j. To verify the observation, note first that for each i, $I_i \supseteq I$. Thus, if I_i is in U_j and U_k, $j \neq k$, then I_i contains a square $K(2, 2)$ generated by points a, b in C_j, and points c, d in C_k. But it is easy to verify that indifference graphs cannot contain squares as subgraphs (cf. [4]). Conversely, if I_i is in none of the U_j, then since $I_i \supseteq I$, it is complete. It follows that $G = \bigcap_{r \neq i} G_r$, and so cub $G < n$.

Since each I_i is in one and only one U_j, it follows that $\sum_{n_j > 1} |U_j| = n$, where $|\cdot|$ denotes cardinality. Since cub $H_j =$ cub $K_j \leq |U_j|$ if $n_j > 1$, and cub $H_j =$ cub $S(1) = 0$ if $n_j = 1$, we have

$$\sum \text{cub } S(n_j) = \sum \text{cub } H_j = \sum_{n_j > 1} \text{cub } H_j \leq \sum_{n_j > 1} |U_j| = n = \text{cub } G.$$

Thus, $\sum$ cub $S(n_j) \leq$ cub G, and the theorem is proved.

It should be remarked that the same argument establishes a more general theorem, of which Theorem 2 is a special case. To state this, we need a notion which will be useful later. A point x in a graph (A, I) is a *focal point* if xIa, all $a \in A$. If G is a graph, we shall denote by G^f, its *focalization*, the graph obtained by adjoining a focal point. Thus, for example, if G consists of n points with no lines, G^f is $S(n)$. We have here the following theorem:

THEOREM 3. *Let $G = (A, I)$ be a graph. Suppose A can be written as the disjoint union of $A_1, A_2, \ldots, A_p$, where $p > 1$, and suppose that in G, all lines between points in different classes A_j occur. Then, if G_j denotes the subgraph generated by A_j, we have* cub $G = \sum$ cub G_j^f.

We are now in a position to calculate $d(n)$, the maximum cubicity of all complete partite graphs with n points, while leaving for the next section the proof that $c(n) = d(n)$. It is not too hard to see, using Theorems 1 and 2, that given n, we maximize the cubicity of $K(n_1, n_2 \ldots, n_p)$, with $n_1 + n_2 + \cdots + n_p = n$, if we take as many of the n_j as possible equal to 3, and the remaining one as 1 or 2 if necessary. Thus, for $n = 3k + i$, $0 \leq i \leq 2$, we have $d(n) =$ cub $K(3, 3, \ldots, 3, i)$, with k 3's. Using Theorems 1 and 2, we can show:

THEOREM 4. *$d(n) = [\frac{2}{3}n]$ if $n \neq 3$ and $d(3) = 1$.*

COROLLARY 4.1. If $n \geq 4$ and $n \neq 6$, $d(n) = d(n - 3) + 2$.

To close this section, we note a simple characterization of the complete partite graphs, which is useful in showing that $c(n) = d(n)$, and whose proof is left to the reader.

LEMMA 1. *A graph is complete partite if and only if it does not contain a subgraph of the form*

$$\begin{array}{ccc} b \cdot\!\!\! & \text{———————} & \!\!\!\cdot\, c, \\ & \dot{a} & \end{array} \tag{5}$$

i.e., if and only if there are 3 distinct points a, b, c so that bIc but neither aIb nor aIc.

4. CALCULATION OF THE MAXIMUM CUBICITY

In this section, we present the proof of the following theorem.

THEOREM 5. $c(n) = d(n)$ for *all* n.

To begin with, we reduce the problem to the study of graphs which have focal points. If B is a set of points of the graph G, $G - B$ denotes the subgraph generated by points not in B.

LEMMA 2. *If a graph* $G = (A, I)$ *has a subgraph of the form* (5), *then* $\text{cub}\, G \leq \text{cub}(G - \{a, b, c\})^f + 2$.

PROOF: We note first that $G = H \cap K$, where H is the graph obtained from G by adjoining for each $x \in A$ (including a, b, c) the lines between x and a, b, c; and K is the graph obtained from G by adjoining all lines between points in $A - \{a, b, c\}$. It follows that $\text{cub}\, G \leq \text{cub}\, H + \text{cub}\, K$. Now, in H, the points a, b, and c are all equivalent and each is a focal point. Thus, H has the same cubicity as the graph $G - \{a, b, c\}$ with a focal point adjoined, i.e., $\text{cub}\, H = \text{cub}(G - \{a, b, c\})^f$. It is left to show that $\text{cub}\, K \leq 2$. To prove this, we write down an explicit embedding f_1, f_2 of K into 2-space. Let

$$\begin{array}{lll} f_1(a) = 0, & f_1(b) = 2, & f_1(c) = 3/2, \\ f_2(a) = 0 & f_2(b) = 3/2, & f_2(c) = 2; \end{array}$$

and if $x \neq a, b, c$, define f_1 and f_2 according to Table 1. We leave it to the reader to check that f_1, f_2 actually do embed K into 2-space.

Now let $e(n)$ denote the maximum cubicity of all graphs with $n + 1$ points, including a focal point, or alternatively the maximum cubicity of the focalization of a graph with n points. Then, we have

LEMMA 3. *If* $n \neq 3$, $e(n) \leq d(n)$.

PROOF: The major step in the proof is to establish for $n \geq 4$ the recursion inequality:

$$e(n) \leq \max\{d(n), \operatorname{cub} S(n), e(n-3)+2\}. \tag{6}$$

To prove (6), suppose G has $n+1$ points, including a focal point x. If $G-x$ is complete partite, then $G-x = K(n_1, n_2, \ldots, n_p)$ and $G = K(1, n_1, n_2, \ldots, n_p)$. By Theorem 2, if $p > 1$, we have $\operatorname{cub} G = \operatorname{cub} G - x$, which is less than or equal to $d(n)$. If $p = 1$, then $n_p = n$, $G = S(n)$, and $\operatorname{cub} G = \operatorname{cub} S(n)$. To establish (6), it is now sufficient to assume that $G-x$ is not complete partite and to prove that $\operatorname{cub} G \leq e(n-3)+2$. If $G-x$ is not complete partite, then $G-x$ has a subgraph of the form (5). Hence, (5) is a subgraph of G as well, and $x \neq a, b, c$. By Lemma 2, $\operatorname{cub} G \leq \operatorname{cub}(G-\{a,b,c\})^f + 2$. But $(G-\{a,b,c\})^f$ consists of a focal point y added to $G-\{a,b,c\}$. Now, $x \in G-\{a,b,c\}$ and thus x and y are two distinct focal points of $(G-\{a,b,c\})^f$. It follows that x and y are equivalent here, and so $\operatorname{cub}(G-\{a,b,c\})^f = \operatorname{cub}(G-\{a,b,c,x\})^f$. But $G-\{a,b,c,x\}$ has $n-3$ points, and so $\operatorname{cub}(G-\{a,b,c,x\})^f \leq e(n-3)$. Thus $\operatorname{cub} G \leq \operatorname{cub}(G-\{a,b,c\})^f + 2 \leq e(n-3)+2$. This establishes (6).

If $n \geq 4$, then as a final preliminary it is easy to verify that $\operatorname{cub} S(n) \leq d(n)$, and so we get from (6) the simpler inequality

$$e(n) \leq \max\{d(n), e(n-3)+2\}. \tag{7}$$

The lemma is easily established for $n = 1, 2$; for $e(1) = d(1) = 0$, $e(2) = d(2) = 1$. Note next that $e(3) = 2$. For, $\operatorname{cub} S(3) = 2$ and, as the reader can readily verify for himself, all 4-point graphs with focal points are embeddable into 2-space. The lemma for $n = 4, 5, 6$ now follows by a calculation using (7). Finally, for $n \geq 7$, by way of induction, $e(n-3)+2 \leq d(n-3)+2$, which by Corollary 4.1 is $d(n)$. Thus, $e(n) \leq \max\{d(n), e(n-3)+2\} = d(n)$.

TABLE I

If x is adjacent to	$f_1(x)$	$f_2(x)$
none of a, b, c	7/4	1/4
a only (among a, b, c)	1	1/4
b only	3/2	1/2
c only	3/4	5/4
a, b only	1	1/2
a, c only	3/4	1
b, c only	7/4	1
a, b, c	1	1

We are now ready to complete the proof of Theorem 5. Suppose first $n \neq 3, 6$, and suppose the graph G has n points. It is of course sufficient to prove cub $G \leq d(n)$. This is trivial if G is complete partite. Otherwise, G has a subgraph of the form (5) and so by Lemma 2, cub $G \leq \text{cub}(G - \{a, b, c\})^f + 2$. Clearly $n \geq 4$, since $n \neq 3$. Now $G - \{a, b, c\}$ has $n - 3$ points, so $\text{cub}(G - \{a, b, c\})^f \leq e(n-3)$. It follows that cub $G \leq e(n-3) + 2$, which by Lemma 3 and Corollary 4.1 is less than or equal to $d(n-3) + 2 = d(n)$, since $n \geq 4$ and $n \neq 6$. This completes the proof of Theorem 5 in the case $n \neq 3, 6$.

It is simple to verify that $c(3) = d(3) = 1$, thus settling the case $n = 3$. If $n = 6$, let x be an arbitrary point of G and note that $G = H \cap K$, where H is obtained from G by adjoining lines between x and all other points, and K is obtained from G by adjoining all lines between points of $A - \{x\}$. Thus, cub $G \leq$ cub $H +$ cub K. It is easy to show that cub $K \leq 1$, while cub $H \leq e(5) \leq d(5)$. Thus, cub $G \leq d(5) + 1 = d(6)$, and this completes the proof of Theorem 5.

5. The Boxicity

We turn now to the generalization of the notion of interval graph and ask whether a graph (A, I) is representable as the *intersection graph* [in the sense of Eq. (1)] of boxes in E^n. Many of the results of Sections 2–4 go over if we define the *boxicity* of a graph G, box G, as the smallest n such that G is representable as the intersection graph of boxes in E^n. As before, we take box $G = 0$ iff G is complete. We sketch the results here. Note first that by projecting into the coordinate axes, it is simple to prove that a graph G is representable as the intersection graph of boxes in E^n if and only if G is the intersection of n interval graphs. Thus, box $G \leq$ cub G, since each indifference graph is trivially an interval graph. Hence, each graph has finite boxicity. Also note that if $G = H \cap K$, then box $G \leq$ box $H +$ box K. Finally, if aEb, then box $G =$ box $G - a =$ box $G - b$.

Theorem 6.

$$\text{box } S(n) = \begin{cases} 1 & \text{if} \quad n > 1 \\ 0 & \text{if} \quad n = 1. \end{cases}$$

Proof: $S(n)$ is an interval graph.

Theorem 7. *box* $K(n_1, n_2, \ldots, n_p) = \sum \text{box } S(n_j) =$ *the number of* n_j *which are bigger than* 1.

PROOF: If $p > 1$, the proof of Theorem 2 applies almost verbatim. Otherwise, the result is trivial.

Note that if all n_j are greater than 1, the inequality "less than or equal to" in Theorem 7 also can be proved by taking p families of parallel $(p-1)$-dimensional hyperplanes in p-space. This naive representation turns out to be optimal. Suppose now $C(n)$, $D(n)$, and $E(n)$ are defined for boxicity analogously to $c(n)$, $d(n)$, and $e(n)$ for cubicity. Then we have

COROLLARY 7.1. $D(n) = [n/2]$ *for all* n.

COROLLARY 7.2. $D(n) = D(n-2) + 1$ *for all* $n \geq 3$.

LEMMA 4. *Suppose* $G = (A, I)$ *is a graph,* a, $b \in A$, *and* $\sim aIb$. *Then,* $\text{box } G \leq \text{box}(G - \{a, b\})^f + 1$.

PROOF: We write $G = H \cap K$, where H is obtained from G by adjoining for each $x \in A$ (including a, b) the lines between x and the points a and b; and K is obtained from G by adjoining all lines between points of $A - \{a, b\}$. We note $\text{box } G \leq \text{box } H + \text{box } K$. Moreover, in H, a and b are equivalent and each is a focal point, so $\text{box } H = \text{box}(G - \{a, b\})^f$. Finally, $\text{box } K \leq 1$, i.e., K is an interval graph. This is easy to see by the characterization of Lekkerkerker–Boland [3].

LEMMA 5. $E(n) \leq D(n)$ *for all* n.

PROOF: We first show that if $n \geq 3$, then $E(n) \leq E(n-2) + 1$. For, let G have $n+1$ points and a focal point, x. If G is complete, then $\text{box } G = 0 \leq E(n-2) + 1$. Otherwise, there are a, $b \in G$ so that $\sim aIb$. Note that a, $b \neq x$ so $x \in G - \{a, b\}$. By the previous lemma, $\text{box } G \leq \text{box } (G - \{a, b\})^f + 1$. But the focal point added to $G - \{a, b\}$ in $(G - \{a, b\})^f$ is equivalent to x, so $\text{box}(G - \{a, b\})^f = \text{box}(G - \{a, b, x\})^f$. Now $G - \{a, b, x\}$ has $n-2$ points, and therefore $\text{box}(G - \{a, b, x\})^f \leq E(n-2)$. Thus, $\text{box } G \leq E(n-2) + 1$, establishing the desired inequality.

The lemma follows by induction and Corollary 7.2. For, $E(1) = D(1) = 0$, $E(2) = D(2) = 1$, and $E(n) \leq E(n-2) + 1 \leq D(n-2) + 1 = D(n)$.

THEOREM 8. $C(n) = D(n)$ *for all* n.

PROOF: $C(1) = D(1) = 0$, $C(2) = D(2) = 1$. Suppose G has $n \geq 3$ points. If G is complete, then $\text{box } G = 0 \leq D(n)$. Otherwise, by Lemma 4, $\text{box } G \leq$

$E(n-2)+1$, which is less than or equal to $D(n-2)+1 = D(n)$ by Lemma 5 and Corollary 7.2. Thus, $C(n) \leq D(n)$.

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References

1. D. R. Fulkerson and O. A. Gross, Incidence Matrices and Interval Graphs, *Pacific J. Math.* **15** (1965), 835–855.
2. P. C. Gilmore and A. J. Hoffman, A Characterization of Comparability Graphs and of Interval Graphs, *Canad. J. Math.* **16** (1964), 539–548.
3. C. G. Lekkerkerker and J. C. Boland, Representation of a Finite Graph by a Set of Intervals on the Real Line, *Fund. Math.* **51** (1962), 45–64.
4. F. S. Roberts, Indifference Graphs, in *Proof Techniques in Graph Theory* (*Proc. 2nd Ann Arbor Graph Theory Conf.*, 1968) (F. Harary, ed.), Academic Press, New York, 1969.